

Mukhanov-Sasaki equations in Hybrid Loop Quantum Cosmology

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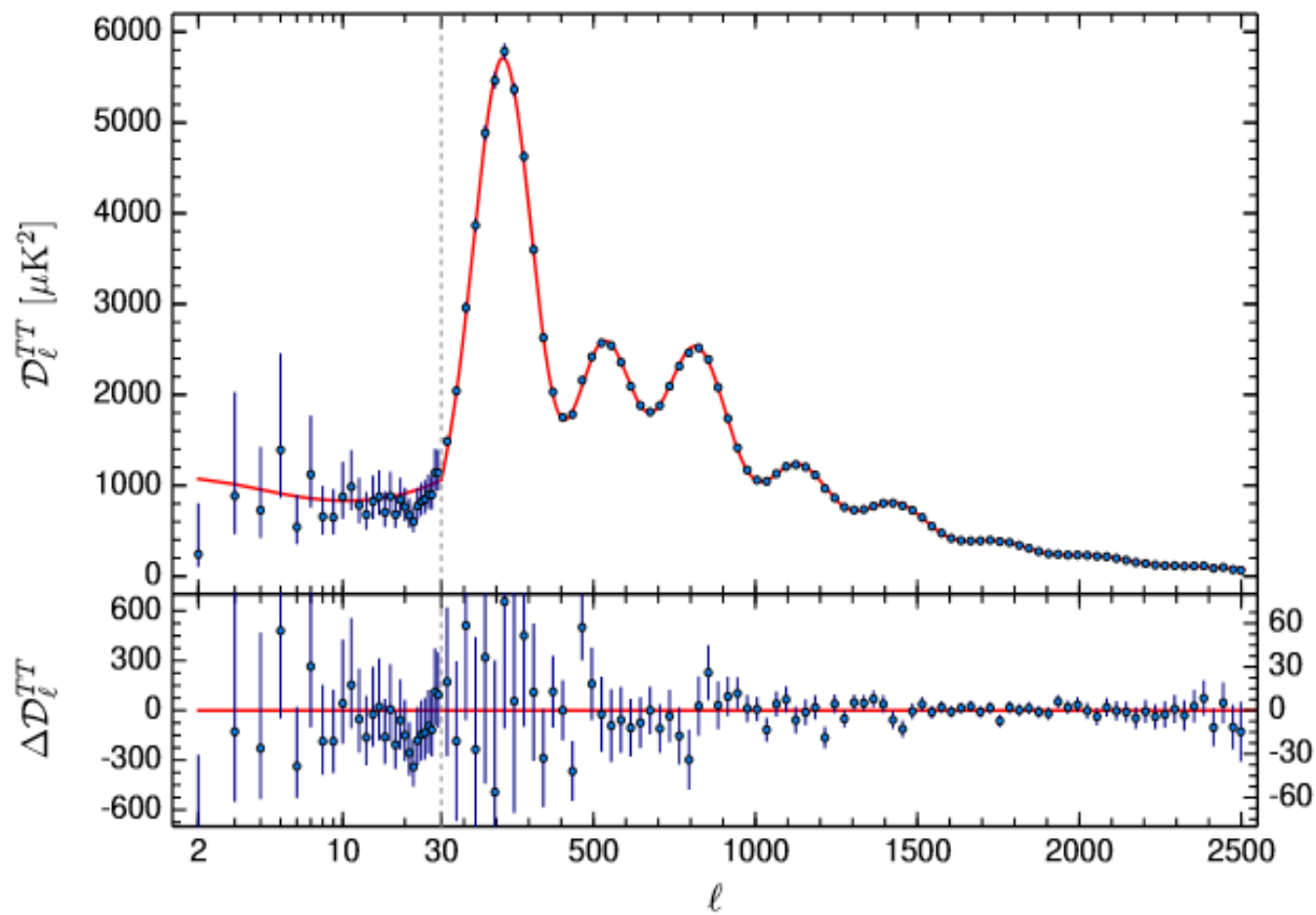
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The theory of **cosmological perturbations**, combined with the **inflationary** paradigm, conciliates the theoretical models of the early universe with observations:

- ✓ Anisotropies of the cosmic microwave background (CMB)
- ✓ Formation of structures at large scales

Need for a quantum theory:

- Very quantum nature of the perturbations
- Singularity-free and consistent description of the evolution of the universe

Planck 2015 results. XIII. Cosmological parameters

Investigate whether it is possible to find information about the **quantum nature of the geometry** of spacetime **in the quantum fluctuations** in the early universe.

- Quantization of **inhomogeneous** cosmological systems.

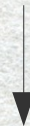
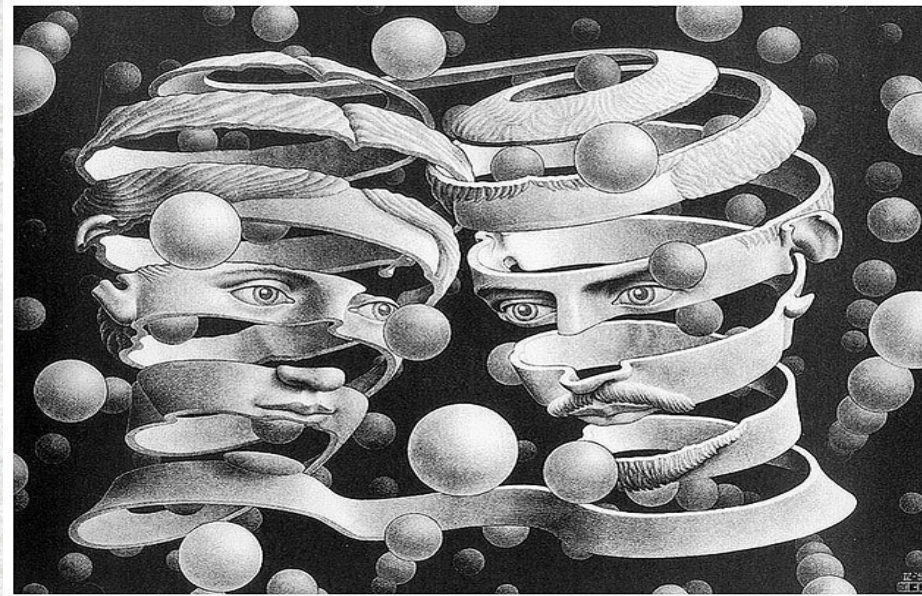
- Strategy:

PHASE SPACE

- Homogeneous sector → Loop Quantum Cosmology
- Inhomogeneous sector → Fock quantization

- Assumption: main (loop) quantum effects on geometry are those concerning the homogeneous degrees of freedom of the background.

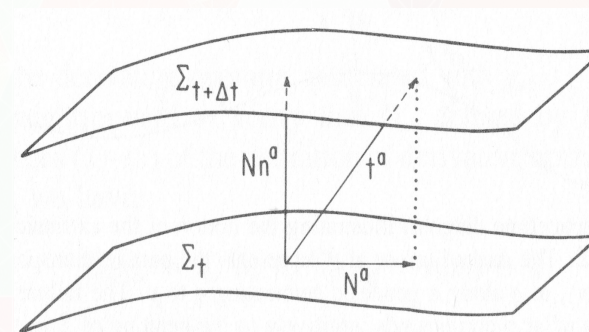
QUANTUM GRAVITY



QUANTUM FIELD THEORY
on curved spacetime

Addresses the quantization of cosmological systems following the ideas and techniques of Loop Quantum Gravity, a **canonical**, **non-perturbative** and **background independent** program for the quantization of general relativity (GR):

- Starting from a Hamiltonian formalism of GR.



- Classical phase space: **Ashtekar-Barbero variables**.

$$E_i^a = \sqrt{\hbar} e_i^a; \quad A_a^i = \Gamma_a^i + \gamma K_a^i \quad \longrightarrow \quad \{A_i^a(x), E_b^j(y)\} = 8\pi G \gamma \delta_b^a \delta_i^j \delta^3(x-y)$$

★ Triads allow the coupling of fermions.

- Dirac's canonical quantization programme for constrained systems.

Avoids the Big Bang



Scalar perturbations in:

FLRW universe + massive coupled scalar field

Approximation:

Truncation at quadratic perturbative order in the action

- Compact flat spatial topology T^3 .
- Scalar field is subject to a quadratic potential.

- Mode expansion of the inhomogeneities: metric and field.
- Adopt the real modes of the Laplace-Beltrami operator compatible with the metric.
- We call $g_{\vec{n},\pm}$ and $k_{\vec{n},\pm}$ the (properly scaled) Fourier coefficients of the lapse function and shift vector.
- Truncated action at quadratic order:

$$H = N_0 \left[H_{|0} + \sum H_{|2}^{\vec{n},\pm} \right] + \sum g_{\vec{n},\pm} \tilde{H}_{|1}^{\vec{n},\pm} + \sum k_{\vec{n},\pm} \tilde{H}_{-1}^{\vec{n},\pm}$$

- Canonical transformation for perturbations:

$$\{X_l^{\vec{n},\pm}\} \equiv \{a_{\vec{n},\pm}, b_{\vec{n},\pm}, f_{\vec{n},\pm}; \pi_{a_{\vec{n},\pm}}, \pi_{b_{\vec{n},\pm}}, \pi_{f_{\vec{n},\pm}}\} \longrightarrow \{V_l^{\vec{n},\pm}\} \equiv \{\mathbf{v}_{\vec{n},\pm}, C_{|1}^{\vec{n},\pm}, C_{-1}^{\vec{n},\pm}; \pi_{\mathbf{v}_{\vec{n},\pm}}, \check{H}_{|1}^{\vec{n},\pm}, \tilde{H}_{-1}^{\vec{n},\pm}\}$$

- Canonical transformation for homogeneous sector:

$$\{q_A; \pi_{q_A}\} \equiv \{\alpha, \phi; \pi_\alpha, \pi_\phi\} \longrightarrow \{\tilde{q}_A; \tilde{\pi}_{q_A}\} \equiv \{\tilde{\alpha}, \tilde{\phi}; \tilde{\pi}_\alpha, \tilde{\pi}_\phi\}$$

Hamiltonian :

$$H = \bar{N}_0 \left[H_{|0} + \sum \check{H}_{|2}^{\vec{n},\pm} \right] + \sum G_{\vec{n},\pm} \check{H}_{|1}^{\vec{n},\pm} + \sum K_{\vec{n},\pm} \tilde{H}_{-1}^{\vec{n},\pm}$$

$$\check{H}_{|2}^{\vec{n},\pm} = \frac{e^{-\tilde{\alpha}}}{2} \left[\omega_n^2 + e^{-4\tilde{\alpha}} \pi_{\tilde{\alpha}}^2 + \tilde{m}^2 e^{2\tilde{\alpha}} \left(1 + 15\tilde{\phi}^2 - 12\tilde{\phi} \frac{\pi_{\tilde{\phi}}}{\pi_{\tilde{\alpha}}} - 18\tilde{m}^2 e^{6\tilde{\alpha}} \frac{\tilde{\phi}^4}{\pi_{\tilde{\alpha}}^2} \right) \left(v_{\vec{n},\pm}^2 + (\pi_{\mathbf{v}_{\vec{n},\pm}})^2 \right) \right]$$

MS Hamiltonian

- Quantum representation of the constraints:

Linear perturbative constraints are represented by momenta operators that act as derivatives.

- Physical states only depend on the homogeneous variables and the MS gauge invariants

Hilbert space: $H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt} \otimes \mathcal{F}.$

- We must still impose the **scalar constraint** given by: $e^{3\tilde{\alpha}}(H_{|0} + \check{H}_{|2}) = 0$

- Hybrid quantization: Fock representation for the inhomogeneous sector

Annihilation and creation operators for our (rescaled) MS gauge invariants and naturally associated with the massless scalar field.

Massive scalar field minimally coupled to a compact, flat FLRW universe:

Geometry: $A_a^i = c^0 e_a^i (2\pi)^{-1}; \quad E_i^a = p \sqrt{0} e^0 e_i^a (2\pi)^{-2} \quad \{c, p\} = 8\pi G \gamma / 3$

⇒ $a^2 = e^{2\alpha} = |p| (2\pi \sigma)^{-2}; \quad \pi_\alpha = -pc (\gamma 8\pi^3 \sigma^2)^{-1}$

Matter: $\phi = (2\pi)^{3/2} \sigma \varphi; \quad \pi_\phi = (2\pi)^{-3/2} \sigma^{-1} \pi_\varphi$

Hamiltonian: $C_0 = -\frac{6}{\gamma^2} \sqrt{|p|} c^2 + \frac{8\pi G}{V} (\pi_\varphi^2 + m^2 V^2 \varphi^2)$

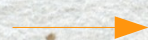
$\sigma^2 = G (6\pi^2)^{-1}$
 $V = |p|^{3/2}$

QUANTIZATION: HOMOGENEOUS SECTOR

The kinematic **Hilbert space** is $H_{kin}^{FRW-LQC} \otimes H_{kin}^{matt}$.

For the FLRW-geometry sector:

- Adopt a basis of **volume eigenstates** $\{|v\rangle; v \in \mathbb{R}\}$, with $\hat{v} \propto |\hat{p}|^{3/2}$.
- The inner product is **discrete**: $\forall v_1, v_2 \in \mathbb{R}, \langle v_1 | v_2 \rangle = \delta_{v_1}^{v_2}$.
- On straight edges, holonomy elements are linear in $N_{\bar{\mu}} := e^{i\bar{\mu}c/2}$.
- We adhere to the so-called **improved dynamics** prescription of LQC



Δ , minimum non-vanishing area

$$\hat{N}_{\bar{\mu}}|v\rangle := |v+1\rangle, \quad \hat{v}|v\rangle = v|v\rangle$$

QUANTIZATION: HOMOGENEOUS SECTOR

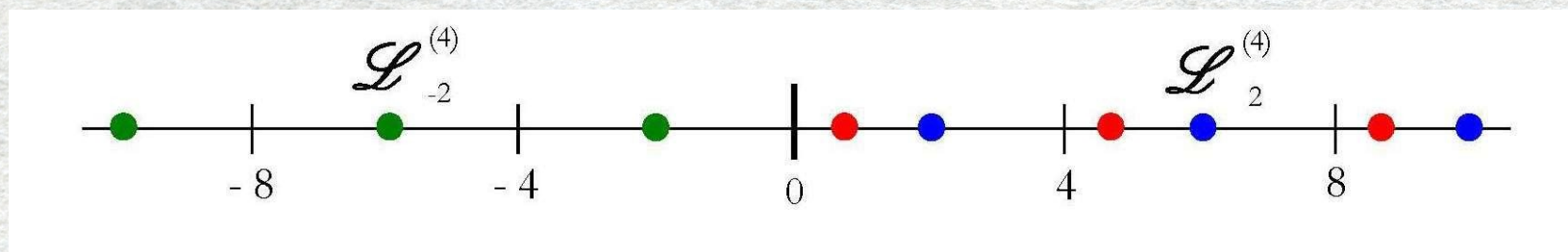
- The inverse volume is regularized as usual in LQC.
- After **decoupling the zero-volume** state, we change the constraint densitization

$$\hat{C}_0 = \left[\widehat{\frac{1}{V}} \right]^{1/2} \hat{C}_0 \left[\widehat{\frac{1}{V}} \right]^{1/2} \quad \hat{C}_0 = -\frac{3}{4\pi G \gamma^2} \hat{\Omega}_0^2 + \hat{\pi}_\varphi^2 + m^2 \hat{\varphi}^2 \hat{V}^2$$

- With **our proposal**, the gravitational part is a difference operator:

$$\hat{\Omega}_0^2 |v\rangle = f_+(v) |v+4\rangle + f(v) |v\rangle + f_-(v) |v-4\rangle$$

that acts on the **superselection sectors** $\mathcal{L}_{\pm\epsilon}^{(4)} := \{ \pm (\epsilon + 4n), n \in \mathbb{N} \}, \epsilon \in (0, 4]$.



- To quantize the quadratic contribution of the perturbations to the Hamiltonian we adapt the **proposals of the homogeneous sector** and use a symmetric factor ordering:

- ➔ **Symmetrized products** of the type $f(\hat{\Phi})\hat{\pi}_{\Phi}$.
- ➔ **Symmetric geometric** factor ordering $V^k A \rightarrow \hat{V}^{k/2} \hat{A} \hat{V}^{k/2}$.
- ➔ Adopting the LQC representation $(cp)^{2m} \rightarrow [\hat{\Omega}_0^2]^m$.
- ➔ In order to preserve the FLRW **superselection sectors**, we adopt the prescription $(cp)^{2m+1} \rightarrow [\hat{\Omega}_0^2]^{m/2} \hat{\Lambda}_0 [\hat{\Omega}_0^2]^{m/2}$, where $\hat{\Lambda}_0$ is defined like $\hat{\Omega}_0$ but with double steps.

- The Hamiltonian constraint reads
$$\hat{C}_0 - \sum \hat{\Theta}_e^{\vec{n}, \pm} - \sum (\hat{\Theta}_o^{\vec{n}, \pm} \hat{\pi}_{\Phi})_{sym} = 0.$$

- Consider states whose evolution in the inhomogeneities and the FRW geometry presents different rates of variation:

$$\Psi = \chi_0(\tilde{\alpha}, \tilde{\phi}) \psi(N, \tilde{\phi}), \quad \chi_0(\tilde{\alpha}, \tilde{\phi}) = \mathbf{P} \left[\exp \left(i \int_{\tilde{\phi}_0}^{\tilde{\phi}} d\phi \hat{H}_0(\phi) \right) \right] \chi_0(\tilde{\alpha}).$$

- The FRW state is peaked and evolves unitarily.

- Disregard **nondiagonal** elements for the FRW geometry sector in the constraint.



$$-\partial_{\tilde{\phi}}^2 \psi - i \left(2 \langle \hat{H}_0 \rangle_{\chi} - \langle \hat{\Theta}_o \rangle_{\chi} \right) \partial_{\tilde{\phi}} \psi = \left[\langle \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0)_{sym} \rangle_{\chi} + i \langle \mathbf{d}_{\tilde{\phi}} \hat{H}_0 - \frac{1}{2} \mathbf{d}_{\tilde{\phi}} \hat{\Theta}_o \rangle_{\chi} \right] \psi$$

where $\mathbf{d}_{\tilde{\phi}} \hat{O} = \partial_{\tilde{\phi}} \hat{O} - i [\hat{H}_0, \hat{O}]$.

- Consider states whose evolution in the inhomogeneities and the FRW geometry presents different rates of variation:

$$\Psi = \chi_0(\tilde{\alpha}, \tilde{\phi}) \psi(N, \tilde{\phi}), \quad \chi_0(\tilde{\alpha}, \tilde{\phi}) = \mathbf{P} \left[\exp \left(i \int_{\tilde{\phi}_0}^{\tilde{\phi}} d\phi \hat{H}_0(\phi) \right) \right] \chi_0(\tilde{\alpha}).$$

➤ The FRW state is peaked and evolves unitarily.

- Disregard **nondiagonal** elements for the FRW geometry sector in the constraint.
- If we can **neglect** :
 - The second derivative of ψ ,
 - The total $\tilde{\phi}$ -derivative of $2\hat{H}_0 - \hat{\Theta}_o$,

$$-i \partial_{\tilde{\phi}} \psi = \frac{\langle \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0)_{sym} \rangle_{\chi}}{2 \langle \hat{H}_0 \rangle_{\chi}} \psi$$

**Schrödinger-like
equation**

- There are **restrictions** on the range of validity.
- The extra terms are negligible if so are the Φ -derivatives of

$$\langle \hat{H}_0 \rangle_\chi, \langle \hat{\Theta}_e \rangle_\chi, \langle \hat{\Theta}_o \rangle_\chi, \langle (\hat{H}_0 \hat{\Theta}_o)_{\text{sym}} \rangle_\chi.$$

- These derivatives contain two types of terms. One comes from the explicit dependence, and is proportional to powers of the **mass**.
- The other comes from commutators with \hat{H}_0 in the FLRW geometry.
- Contributions arising from $[\hat{\Omega}_0^2, \hat{V}]$ can be relevant. In the effective regime.

- Starting from the **Born-Oppenheimer** form of the constraint and assuming a direct **effective** counterpart for the **inhomogeneities**:

$$d_{\eta_\chi}^2 v_{\vec{n}, \pm} = -v_{\vec{n}, \pm} 4\pi \left[\omega_n^2 + \frac{\langle \hat{\theta}_e^q + (\hat{\theta}_o \hat{H}_0)_{sym} - \frac{i}{2} \mathbf{d}_{\tilde{\phi}} \hat{\theta}_o \rangle_\chi}{\langle \hat{\theta}_e \rangle_\chi} \right]$$

where

$$\hat{\theta}_o = \sqrt{\frac{12G}{\pi}} 2\gamma m^2 \hat{\phi} \hat{V}^{2/3} |\hat{\Omega}_0|^{-1} \hat{\Lambda}_0 |\hat{\Omega}_0|^{-1} \hat{V}^{2/3},$$

$$\hat{\theta}_e = \frac{3}{2G} \hat{V}^{2/3},$$

$$\hat{\theta}_e^q = \frac{2G}{3} \left[\frac{\hat{1}}{\hat{V}} \right]^{1/3} \hat{H}_0^{(2)} \left(19 - 32\pi^2 G^2 \gamma^2 \hat{\Omega}_0^{-2} \hat{H}_0^{(2)} \right) \left[\frac{\hat{1}}{\hat{V}} \right]^{1/3} + \frac{3m^2}{8\pi^2 G} \hat{V}^{4/3} \left(1 - \frac{8\pi G}{3} \hat{\phi}^2 \right).$$

CONCLUSIONS

- We have presented the **hybrid loop quantization** of a FLRW universe with a massive scalar field perturbed at **quadratic** order in the action.
- The system is a **constrained symplectic manifold**. **Backreaction** is included at the considered truncation order.
- The model has been described in terms of the Mukhanov-Sasaki **gauge-invariant**.
- A **Born-Oppenheimer** ansatz leads to a Schrödinger equation for the inhomogeneities. We have discussed the range of validity.
- Finally, we have derived the effective **Mukhanov-Sasaki equations**. The ultraviolet regime is **hyperbolic**.

